an infinitesimal distance from the circumference, then the conjugate curve to the resultant extremely narrow hyperbola, or pair of parabolas, being extremely broad, must have a major axis and director circle of infinitesimal magnitude; strictly corresponding, then, in its resultant ratios to one immeasurable distant. While as the fixed point of the original curve recedes from its circle, so reciprocally do the director circles of its conjugate curve grow in proportionate magnitude; and the comparative distance of their respective points decrease. Until, said fixed point of the parent curve having passed to infinity, and its hyperbola therefore, as already stated, degenerated into a right line, perpendicular to the major axis, the conjugate curve must now possess director circles infinite in radius, and infinitesimally close to their respective fixed points; and thus be, first, momentarily, a pair of parabolas, and then a similar pair of degenerated right lines, lying parallel to, and indeed inside of, and coalescing with, the degenerated original curve. At the one extreme, then, of this sequence that we have traced, lies the circle, with its foci coalesced into the central pole. And at the opposite extreme, as the farthest possible removed form, lies the degenerated "perpendicular right line hyperbola," with its infinitely distant fixed point on the polar of said pole. While midway lie the two sets of parabolas, facing in opposite directions, and each flashing momentarily into being, as the ellipse sinks into, or the hyperbola arises from their medial point of indifference; which is, again, a degenerated horizontal right line.

From the above the following theorems can be deduced:

Theorem 1. Even as the sum, in the ellipse, or difference, in the hyperbola, of the two focal distances of any point P on the curve is constant, and equal to the major axis; so, similarly, is the sum or difference of its distances from the two director circles.

Theorem 2. The line SE or fE, SH or fH, etc., joining the fixed point S or f to the extremity of any radius $S'E$ or $S'H$ of its director circle, is ever bisected by the auxiliary circle to the curve. While if the two radii, in the ellipse, or diameters in the hyperbola, of the two director circles be drawn through the same point P or p on the curve, and their extremities joined with their respective fixed points [S or f, as the case may be]; and the bisections of such two lines, by the auxiliary circle, be joined by a right line, then will this line be the tangent at P or p.

Theorem 3. In the ellipse, the produced latus rectum, which is the double sine through f, the fixed point, will intersect upon the director circle those two radii, which determine the minor axis of the curve. For if $S'B=Bf=BH$ [Figure 1], then $S'fH$ are concyclic; $S'fH$ a right angle; and thus fH the produced semi-latus rectum.

Theorem 4. While correspondingly, in the hyperbola, those two diameters of the director circle, whose respective tangents pass through the fixed point, are parallel to the asymptotes, determining at infinity the ideal minor axis of the curve. Which two tangents, furthermore, are perpendicularly bisected by those asymptotes, at their common intersection with the rectilinear directrix.

For, let SH be the radius, whose tangent HS passes through S, the fixed point [Figure 1]. And let ZCZ'', $Z'CZ$ be the asymptotes, meeting the directrix in YY'. Then $SH:SS=AA':SS'=CA:CS=CY:CS$. So that the right-angled triangles SHS and CYS are similar, with SH parallel to the asymptote CYZ; and SY half of HS, even as CS is of SS. And thus CYZ bisects SH at right angles.

Corollary 1. If a perpendicular Cb were raised at C the center of the curve, and through either of the apeci A or A' a line Ab or $A'b$ be drawn, parallel to either SH, or an asymptote, it will cut Cb in b, an extremity of the accepted minor axis of the hyperbola—the major axis of its conjugate.

Theorem 5. If from any point, say K, upon the circumference of the director circle, two tangents KI and KG be drawn to the curve, cutting the circle again in GI, and these points GI be joined; then will said GI be a third tangent to the curve. And the resultant circumscribing triangle KGI will have for its orthocenter S [or f, as the case may be], the fixed point.

Taking the hyperbola [Figure 1]. There are evidently innumerable circumscribing triangles possible. And similarly, innumerable triangles, having S, the fixed point, for their orthocenter. And finally, innumerable triangles which fulfill both of these conditions.

Let one of these latter, KGI, be drawn, both circumscribing the hyperbola, and with its orthocenter at S. We will prove that its vertices KGI lie on the director circle.

The auxiliary circle to the curve cuts the three tangents KI, KG, and GI in points vi, wk, and ug, respectively. Then, by a well known theorem, SiI, SkK, and SgI are right angles; and thus gki are the pedal points of the perpendiculars from the vertices upon the sides of the triangle KGI. And therefore the auxiliary circle, as the circumscribing circle to this pedal triangle kgi, is the "nine points circle" of triangle KGI; having its center C collinear with, and bisecting the distance between the orthocenter S and circumcenter S' of KGI; while its radius is half that of the circumscribing circle, since it is itself the circumscribing circle to the medial triangle uvw. So that since $CS=CS'$, and $CA=\frac{1}{2}AA'=\frac{1}{2}S'D$, therefore the director circle is the circumscribing circle to the triangle KGI, tangential to, and thus circumscribing the hyperbola, and with its orthocenter at the fixed point S.

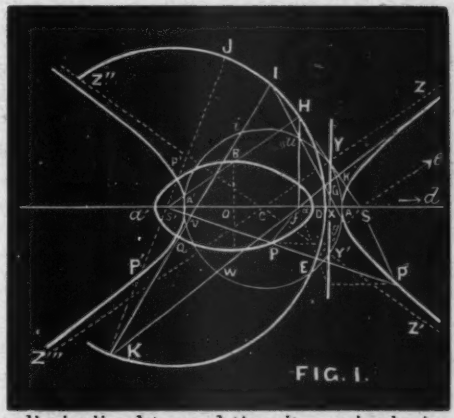
An additional proof is as follows. If uvw be the points where the sides of the circumscribing triangle KGI are cut by its "nine points circle"—the auxiliary circle,—then such points thereby bisect those sides; and the perpendiculars Su, Sv, and Sw upon them must thus meet in S', the circumcenter to the triangle, and second focus to the curve.

And since the same proof holds true, and the auxiliary circle is the "nine points circle" for any triangle, which is both tangential to the hyperbola, and has its orthocenter at S; then, conversely, any two tangents from any point upon the circumference of the director circle must determine a third tangent to the curve; and the resultant circumscribing triangle have its orthocenter at S, the fixed point.

In a like manner, if a triangle be drawn circumscribing an ellipse, and with its orthocenter at f, the fixed point of the curve; then the auxiliary circle is still the "nine points circle," and thus the director circle its circumscribing circle.

In the parabola, since the director circle, being infinite in radius, has become the rectilinear directrix, no real circumscribing triangle, having its orthocenter at the focus S, can be drawn. Although the tangent on the vertex, which represents, of course, the auxiliary circle, would also be the infinite "nine points circle" to any such triangle; since it passes through the pedal points of the perpendiculars from the orthocenter S upon the tangents.

Corollary 1. The auxiliary circle, as the "nine points circle," bisects not only the sides KG, KI, and GI of the circumscribing triangle; but also the three lines SK, SG, and SI, joining its vertices to its orthocenter. While, if the perpendiculars Sg, Sk, and Si be produced, they will ever meet their circumcircle, the director circle, at double the distances Sg, Sk, and Si, respectively.



Corollary 2. The "nine points circle" is thus tangential, not only to the inscribed and escribed circles of triangle KGI, but also to its inscribed ellipse, or escribed hyperbola.

Corollary 3. If the pedal triangle kgi be drawn, then, by known theorems, its sides must be equally inclined to, and thus its angles be internally or externally bisected by the sides of its parent triangle KGI; whose three apeci K, G, and I, and orthocenter S, furthermore, must ever be the centers of its inscribed and escribed circles. While points SkI, SGi and SgK, like KGk, must be ever collinear, in ellipse, or hyperbola. So that points SGI, SGK, and SKI must ever lie on the arcs of circles, whose radii are equal to each other, and to that of the director circle—namely, to the major axis.

But the most important deductions from the foregoing are the following theorems; by means of which the exact opposite and reciprocal reverse, point for point, of any conic curve can be precisely determined. A parabola reversing into an equal, but opposite parabola; a circle into a perpendicular right line; and an ellipse into an exactly reciprocal hyperbola; and conversely; with equal major axi.

Theorem 6. If an ellipse and a hyperbola, with major axi lying collinearly along one right line, and with a common director circle [and thus also with equal, though not coincident, major axi], have their respective fixed points each on the polar of the other, in respect to their common circle, *i. e.*, [Figure 1] $Sf:SD=SD:SS$; then said ellipse and hyperbola will be each the reverse or

reciprocal of the other. The common radii of their common director circle determining corresponding points upon each of the two curves, with focal distances which are equally inclined to their respective major axi.

Thus, if $SpEP$ be a common radius of the common director circle, cutting and determining the ellipse in p, and the hyperbola in P, then will angles pfS, and PSd, or pfa and PSA be equal. Similarly, the same common radius will determine the lati recti in both curves. Or finally, that radius SH, lying parallel to an asymptote of the hyperbola [Theorem 4], thus determining at infinity its ideal minor axis, determines also the minor axis of the reciprocal ellipse [Theorem 3]; and subtends an equal angle in both curves; BfS of the one equalling eSd of the other. And so on, for all the radii.

Taking first the radius SH parallel to the asymptote ZCZ''. Then by construction, $Sf:SH=SH:SS$; so that the triangles SHS and SfH are similar right angled triangles; and fH is thus the produced semi-latus rectum of the ellipse. Therefore SH must pass through and determine B, an extremity of the minor axis of the ellipse [Theorem 3]; even as it also meets the asymptote of the hyperbola at infinity. While by construction, angles HSf, ZCd, and eSd are equal, and thus also BfS and eSd.

Next, taking any other radius $SpEP$; determining the ellipse in p, and the hyperbola in P. Join pf, PS, Ef, and ES. Then again $Sf:SE=SE:SS$; and so SES and SfE are similar triangles, with equal angles opposite to homologous sides, SES to SfE, and SEf to SSE. But pE being equal to pf, and PE to PS, angles pEf and pfE are equal; and also PES to PSE. And therefore angles $pES-pEf$, or fES equal $SfE-pfE$, or pfS. While on the other hand, angles $SSE+PSE$, or PSS equal $SEf+PES$. So that their supplementary angles PSd and fES are equal; and therefore $pfS=PSd$, and $pfa=PSA$.

And similarly, for any radius of the common director circle. So that if the one be taken which determines fl, the semi-latus rectum in the ellipse, and cuts the hyperbola, say, in L; then angle LSd can be shown equal to the right angle lfS; and thus LS be the semi-latus rectum of the hyperbola.

Said ellipse and hyperbola are thus reciprocal curvilinear forms; or the exact reversals and opposites of each other.

But note: when a radius beyond SH is taken, as *e. g.* KSJ, determining therefore upon its *diameter* the reciprocal points p' on the ellipse, and P' on the other branch of the hyperbola, then the axial angle of P', which is equal to $p'fS$ and fJS is, not $P'Sd$, but $P'SS'$. So that the axial angle, in the hyperbola, must thus in every case be measured towards its ideal minor axis, as it is towards the real, in the ellipse.

If in place of f and S, we choose S' for our focal angle, then our theorem still holds true; for obviously, $PS'd=pS'f$, and $P'S'A'=p'S'a'$, etc.

Applying this theorem to the parabola, since it can be considered either an extreme ellipse, or an extreme hyperbola, our reciprocal curves will take the form of two precisely similar parabolas, facing in opposite directions a common directrix. In which case, manifestly the above theorem holds true. While if

we take a circle as our ellipse: then since Sf its inter-focal magnitude equals zero, its polar to f must lie at infinity. And thus its reciprocal hyperbola, or reverse curvilinear form, whose fixed point S is on that polar, must assume, as already pointed out, the degenerate shape of a right line, lying at infinity, perpendicular to the collinear major axi.

If, then, we draw about point S two concentric circles; the larger with double the radius of the smaller, and representing its director circle, and let them be cut in p and E respectively by a radius SpE. Then SpE produced will represent the direction in which, at infinity, lies the ideal point P upon this degenerated right line hyperbola. While any other radius SaD, making less than a right angle with SpE, may represent the collinear major axi; and thus a line $E.....S$, through E, parallel to SaD, will represent the direction at infinity of the ideal fixed point S of this degenerated hyperbola. Then, by hypothesis, $PE=PS$; and thus the ideal angles PSE, PSa, or PSA equal the real angles PES and pSa. So that our theorem is again true.

Corollary 1. Thus if any points p and P upon the reciprocal curves subtend equal angles pfS and PSd, or pfa and PSA; then they are thereby corresponding points, lying on one common determining radius or diameter.

Corollary 2. If pm and PM, their perpendiculars to the directrix, be drawn, then angles fpm and SPM are equal.

Corollary 3. If the lines fE and SE, or fJ and SJ, joining the respective fixed points to the extremity E, or J, of the common determining radius SE, or diameter $P'Sp'J$, be each bisected in r and R by the respective auxiliary circles [Theorem 2], then the line rR, joining said medial points, is thereby fixed, both in direction, and magnitude; being parallel to, and one half of, fS.

Theorem 7. The said reciprocal curves, ellipse and hyperbola, having a common director circle, have thereby also both reciprocal eccentric ratios $Sf:aa'$, or $fa:aX$ of the one, equaling $AA':SS$, or $AX:SA$ of the other; and a common rectilinear directrix YXY'. Which, furthermore, bisects the reciprocal focal distance, *i. e.*, fS. So that $fa:aX$ of the ellipse is not only of equal ratio to its reciprocal $AX:SA$ of the hyperbola; but is also of equal magnitude, fa equalling AX, and aX equalling SA [Figure 1].

For, bisecting fS in X by the perpendicular YXY', then $fD:DS=fD:fS-fD=fS-DS:DS$, and thus their halves are in a similar proportion; $fa:fX-fa=XS-AS:AS$; or $fa:aX=AX:SA$. While $fa+aX=XA+AS$; and so $fa=AX$, and $aX=SA$.

Lastly, if O be the middle point of Sf in the ellipse, and C of SS in the hyperbola, then will $fa:aX=fD:DS=Sf:SD=Sf:aa'=Of:Oa=Of+fa:Oa+aX=Oa:OX$. So that X is on the polar of f in the ellipse; and thus the polar YXY' is the rectilinear directrix.

And similarly, in the hyperbola, $SA:AX=SD:Df=SS:SD=SS=A'A=CS:CA=CS-SA:CA-AX=CA:CX$. So that YXY', as the polar of S, is the rectilinear directrix of the hyperbola.

To draw reciprocal conic curves, then, we may either, with a common

director circle, place their respective fixed points each on the polar of the other; or else, using a common rectilinear directrix, we may make their eccentric ratios reciprocally equal, both in proportion, and in size. The said eccentric ratio $fa:aX=AX:SA$, or polar proportion $fD:DS=Sf:SD=SD:SS$ being the same in both cases. While in either case, also, the major axi are both collinear, and equal; and the second focus S is in common.

Corollary 1. If the focal distance pf of a point p on one of these curves equals in magnitude the distance PM from the directrix of point P on its reciprocal curve, then conversely, must the directrix distance pm of the first point p equal the focal distance PS of the second, P.

Corollary 2. Since $SY=YH$ [Theorem 4], and $SB=Bf=BH$, therefore $YX=OB$. And thus YY', the portion of the directrix cut off between the asymptotes of the hyperbola, ever equals the minor axis of the reciprocal ellipse.

Corollary 3. fX or SX, being the half of fS, equals the distance OC between the respective centers of the curves. For $SS-Sf=fS$; and thus $\frac{1}{2}SS-\frac{1}{2}Sf=\frac{1}{2}fS=fX=SX=SC-SO=OC$. And this distance, again, must equal the space between the several apeci, Aa, or $A'a'$. Since $fa=AX$, and $aX=SA$, and thus $fa+aX=fX=AX+aX=Aa=fa+SA=Sa'+SA'=A'a'$.

Theorem 8. If D be the point where the director circle cuts the major axis of a conic; SE or SJ any radius or diameter of this circle, determining a point P or p on the curve; DE a line joining points D and E; and ES or Ef a line from E to the fixed point S or f. Then will angle DES or DEf, as the case may be, ever be one half of the axial angle of P or p; which, in the ellipse is angle pfS, but in the hyperbola PSd, or PSS' [Theorem 6], according to the branch which point P may be upon [Figure 1].

For in Theorem 6 angles fES, pfS and PSd were shown to be equal; while SfE and SES being similar triangles, the homologous sides $fE:ES=Sf:SE=SE:SS=SD-Sf:SS-SD=fD:DS$. So that DE must bisect angle fES [Euclid VI, 3]; and thus $DEf=\frac{1}{2}pfS$; and $DES=\frac{1}{2}PSd$.

Were $P'Sp'J$ the determining diameter chosen, then in a like manner it can be shown that $fJ:JS=fD:DS$; and thus angles $DJf=DJS=\frac{1}{2}fJS=\frac{1}{2}p'fS=\frac{1}{2}P'SS'$.

In the parabola, PM, the perpendicular through P to the directrix, obviously represents pE or PE as above; and since angles XSM, SMP, and PSM are all equal, SM must bisect angle XSP; and thus SMX, the complement to SMP, be half of PSd, the supplement to XSP.

While, in the circle, making the same construction as in Theorem 6, angles SED, SDE, and DES are all equal; and thus DES or DES is one half of the supplement to $p'Sa$, or to the ideal angle PSa.

NON-EUCLIDEAN SPHERICS.

By DR. GEORGE BRUCE HALSTED.

As part of the general enlightenment springing from the creation of non-Euclidean geometry, every one now knows that spherical trigonometry is entirely independent of the parallel postulate. We wonder that anyone should have stopped to give proof of what is now so obvious.

Yet perhaps the most difficult article in all Lobachevski's "Geometrical Researches on the Theory of Parallels" is §35, which concludes, "Hence spherical trigonometry is not dependent upon whether in a rectilinear triangle the sum of the three angles is equal to two right angles or not."

Just so concludes Chapter XI of his New Elements: "Therefore the equations for spherical triangles remain the same whether we assume the angle of parallelism as constant or variable."

In §26 of Bolyai's Science Absolute of Space spherical trigonometry is established independently of the parallel postulate.

In his non-Euclidean space Bolyai found a uniform surface, F, whose proper geometry is Euclidean, its straight being L, the circle-limit. Lobachevski found the same, calling F orisphere, L oricycle.

But profound as was their genius it never questioned the assumption, of every three costraight points always one and only one is between the other two. As a consequence, the straight was for them of essence unclosed, and space infinite. So the characteristic geometry of the sphere was not given rank with that of the orisphere, and the non-Euclidean geometry of finite space remained unsuspected. Neither reached the conception that the totality of space may be finite.

The circle-limit was infinite and was conceived as the straight of the orisphere; the finite great circle was not conceived as the straight of the sphere. It remained for Riemann to perceive that the straight, though unbounded, need not be infinite, whence followed a new non-Euclidean geometry, now called by his name.

Beltrami showed that in Euclidean space there may be a surface a piece of which may perhaps represent a piece of the Bolyaian plane. Such is the surface of constant negative curvature, the pseudosphere.

True, it is impossible to represent the entire Bolyaian plane on a Beltrami surface without singular points; nevertheless, meaning by pseudospheres surfaces of revolution which have for meridians a tractrix or curve of equal tangents, we may say their characteristic geometry is Bolyaian.

Of late this doctrine has been filled out, completed by the beautiful

THEOREM OF BARBARIN: *Each of the three spaces, Euclidean, Bolyaian, Riemannean, contains surfaces of constant curvature of which the geodesic lines have the metric properties of the straights of the three spaces.*

These surfaces are (1) the tubes or surfaces equidistant from a straight,

it being possible for the distance to be infinite, which gives the orispheres, (characteristic geometry, Euclidean); (2) the pseudospheres (characteristic geometry, Bolyaian); (3) the spheres (characteristic geometry, Riemannean).

In 1879 Killing made clear the distinction between Riemannean space and what he then called its polar form, by Klein called simple elliptic space. This latter, Killing thinks, had been entirely missed by Riemann, as we know it was by Helmholtz even up to 1876 when he still reproduced the old but false theorem that in space of positive curvature two geodetic lines, if they in general intersect, must necessarily intersect in two points. Such a space of constant curvature Klein calls *spherical* in contradistinction to the simple elliptic in which the assumption, two points determine a straight, has no exception.

Killing it was also who first proved that besides the Euclidean, Bolyaian, simple elliptic, the spherical or old Riemannean is the only, the sole space which as a whole can be freely moved in itself. There are undertypes in abundance where the free mobility of figures only holds so long as the dimensions of the figures do not surpass a certain size; a series of topologically distinguishable spaces which for bounded (simply connected) parts are Euclidean, Bolyaian, simple elliptic. Moreover it has been demonstrated, so far as concerns the surfaces of constant positive curvature, to which the Riemannean geometry applies, that apart from the sphere there is no other closed surface of this sort. The sphere is the only closed surface of positive constant curvature without singularities.

All this intensifies the importance of surface spherics, two-dimensional spherics, pure spherics, intrinsic spherics, Riemannean spherics, double-elliptic spherics, non-Euclidean spherics.

Fortunately a place even in general education has been held open for this newcomer. All the theorems of the so-called "solid geometry" of the schools which relate solely to the surface of the sphere, there obtained by using the parallel postulate, by dragging in the globe which in Euclidean geometry is inside the sphere, in fact by considering the sphere as the covering belonging to such a globe, and therefore tri-dimensional, in a three-way-infinite manifold, are really theorems of that simpler finite manifold the sphere, having no dependent relation to the Euclidean straight, plane, or space. How obvious, then, that these theorems should be developed entirely from the assumptions which characterize the sphere.

Even what is ordinarily conceived of as the shape of the sphere is not wholly irrelevant, for, using the terminology of our Euclidean intuition, if the surface or covering of a globe be detached from the globe, any surface into which it can be bent without stretching, into which it can be flexed, is a double elliptic surface, a surface of constant positive curvature with its proper geometry, and what is ordinarily thought of as the free mobility of figures in it may remain; but somewhere on this surface has come a singularity, it is no longer free from singularities.

As an illustration in still lower terms of the meaning of intrinsic properties of the sphere, take the circle, the closed curve which will slide on its trace,

which is mobile in itself as a whole. This occurs in the Euclidean, Bolyaian, Riemannean plane with its intrinsic properties unchanged; but consider its radius, and it flies apart into three. The circumference of a circle in the Euclidean plane equals $2\pi r$; in the sphere the circumference is less than $2\pi r$; in the pseudo-sphere the circumference is greater than $2\pi r$. The intrinsic properties of the sphere are just what we want. Since they are utterly independent of the parallel-postulate, the simplest spherics must be non-Euclidean.

The student's familiarity with the sphere under its old Euclidean aspect as glove to a globe is also an advantage, which to all needing an introduction to the new ways of treating geometries decides in favor of intrinsic spherics as against simple elliptic planimetry with its unilateral plane which we can so strangely get through without going through. In spherics we have familiar material to present in the new light, with the new methods, to be therein acquired that they may be then retained for analogous conquest of unfamiliar realms.

When instead of building up theorems on polyhedral angles and then cutting them back into spherics, we realize the more complex theorems of angloids as already given in the simpler points of the sphere, we appreciate the practical in the theoretic.

How important, how enlightening to set forth the fundamental assumptions which give by pure logic all the relations of spheric figures, and to see developed therefrom the familiar system of theorems which so long constituted spherical geometry.

The old straight line, the old great circle are dissipated, volatilized, and in their place comes the *straightest* to which now applies the old definition of the straight line, "a line which pierces space evenly, so that a piece of space from along one side of it will fit any side of any other portion."

In vulgar phrase, the straightest turns neither to the right nor to the left so far as the sphere is concerned. But motion can never be fundamental, and it is the assumptions which really make the space. There is one geometric entity back even of the straightest, the point. It is the relation of the straightest to the point which differentiates the spheric from the simple elliptic. "The straight," says Mansion, "is a line determined by any two of its points, sufficiently near." But what is the meaning of 'sufficiently near'?

As long ago as 1877, I overcame these difficulties by building up the system of spherical geometry on a set of assumptions expressing only the few fundamental relations of points and straightests. Clearness is subserved by using 'straightest' as the designation for the spheric straight.

Line is a word which has always been used for the genus of which curve is a species, and of late such distractingly, bewilderingly complex curves and lines have appeared, that line in general should have no longer a place in the elements. Point and straightest are consciously accepted as elements to which specific assumptions give the requisite precision. To forestall controversy, one may reserve the word definition to mean an agreement to substitute a simple term or symbol for more complex terms or symbols.

Instead then of Mansion's definition we have what we prefer to call an Assumption of Association: I 1. For every point of the sphere there is always one and only one other point which with the first does not determine a straightest.

This second point we will call the *opposite* of the first. After three more assumptions, we come to a very fundamental yet complex question, the arrangement of one sort of element on the other. The word 'order' is so common that we are not conscious of its complexity.

What is the difference between AB and BA? Does it not involve the assumption of a third, perhaps a fourth something? Of two sounds in time can one come first and the other afterwards unless we assume also the idea of a past? Could there be a present and a future without the idea of a past? To get a future must not the present act as past? Is there any difference between the point-pair AB and the point-pair BA apart from their relation to a bearer, a carrier, be this merely time itself? There being no elements but points, and only three of these, can there be a relation among these three points called 'order'?

If the points ABC and no others are on a bearer, they may be said to have order or no order according as the bearer is open or closed. If they have order it may be ABC with CBA, or ACB with BCA, or BAC with CAB, according to the bearer.

There exists a particular geometry, purely qualitative, making no use of the notion of straightness or planeness, but instead only of those of line and surface. This is the so-called *analysis situs*. Yet here remains order. Is it not then impossible that without loss of generality order should be subjected to straightness? For simplicity then, for certainty, for accuracy and ease, let us come down from the general idea of order to a more specific idea which we intend to apply only to points on a straight or a straightest, and for which we will take as available the unused word 'betweenness.' Hilbert in 1899 stressed the importance of *between* for the arrangement of costraight points. His treatment was extraordinarily simplified by the elegant proof* of my pupil R. L. Moore that one of his assumptions was redundant. But the problem for points on a straightest is far more difficult. Three terms cannot have a cyclic order, and to say of three points on a circle, that each is between the other two is to waste 'between'. So is the inexpert remark that a point, though it does not divide a straightest into two sects, yet makes of it a single piece in which the points are arranged in a natural order; that is, the words "follow," "precede," "lie between" are applicable.

The great working value of betweenness, is that when a point is known to be between two points, it is thereby located on one particular given straightest. But if we accept 'between' in the above inexpert sense, then to say B is between A and C may mean absolutely nothing, since if A and C are opposite, every other point of the sphere is on a straightest with them and, in the inexpert sense, between them. As a consequence, therefore, the very first of our Assumptions

*Monthly, Vol. IX, April, 1902, pp. 100-1. Cf. E. H. Moore, *Transactions*, Vol. 3 (1902), pp. 142-158.

of betweenness on the sphere, to specify how "between" is to be used of points in a straightest on a sphere, must be:

II 1. No point is between two opposites. This is reinforced by

II 2. Between any two points not opposites there is always a third point.

Filling out this scheme, we have a 'between' that can be used, for example in the definition: Two points A and B, not opposites, upon a straightest a, we call a *sect*, and designate it with AB or BA. The points between A and B are said to be points of the sect AB or also situated *within* the sect AB. The remaining points of the straightest a are said to be situated without the sect AB. The points A, B are called *end-points* of the sect AB.

The next set of assumptions are of congruence. This is a matter which the inexpert suppose they can finish briefly thus:

Definition. Geometrical figures which may be carried over into one another by rigid motions are said to be congruent ($\equiv$).

Theorem. A sect is congruent to itself reversed in direction.

Proof. The point A may be be applied to the point B and the direction AB to the direction BA. Then B will fall upon A: for otherwise the part and the whole would be congruent.

But this treatment is entirely inadmissible.

To define congruence by rigid motion is false and fallacious, since the intuition of rigid motion involves, contains, and uses the congruence idea. We must base the idea of motion on the congruence idea.

A man of whom it has been said: "He was by far the most eminent Amerisan of the Colonial Period, whether we regard the influence of his labors and opinions, upon his own time, in his own country, their wide diffusion in others, or that survival of prestige and authority which yet perpetuates his name and memory," Jonathan Edwards, who died president of Princeton, says, "Motion is a body's existing successively in all the immediate contiguous parts of any distance, without continuing for any time in any one of them."

Its geometric substratum, then, is the preexistence of a series of congruent figures. So rigid motion presupposes congruence.

Moreover, we do not need the troublesome idea 'direction.' In the plane, 'same direction' assumes the whole theory of parallels. On the sphere no two straightests have the same direction, since no two are parallel, yet every two have the same direction, since they go from the same point to the same point. Nor is anything gained by agreeing to call a ray a direction. So preceding motion must come congruence, the idea to be made precise by assumptions.

But right here an unexpected and hitherto unsuspected simplification is possible. In his first congruence axiom, III 1, Hilbert explicitly assumes "Every sect is congruent to itself, that is, always $AB \equiv AB$." Now this assumption is redundant, as was Hilbert's II 4. It is a demonstrable theorem. Two proofs of it have been given me by R. L. Moore, based on the Assumptions of congruence:

III 1. If A not$\equiv B$, and A' not$\equiv C'$, then on ray $A'C'$ there is one and only one point, B', such that $AB \equiv A'B'$.

III 2. If $AB \equiv A'B'$ and $A'B' \equiv A''B''$, then $AB \equiv A''B''$.

III 3. If B is between A and C and B' is between A' and C', and $AB \equiv A'B'$ $BC \equiv B'C'$, then $AC \equiv A'C'$.

We next define angle as two rays from a common point, and assume as

III 4: On a given side of a given ray there is one and only one angle congruent to a given angle.

Instead now of two congruence assumptions

III a: Angles congruent to the same are congruent to each other, and

III b: Euclid I. 4,

we may prove these two as theorems by defining congruent angles in terms of congruent sects, and assuming

III 5: Euclid I. 8: If A, B, C are non-costraight and so are A', B', C', and C is between B and D, and C' is between B' and D', and $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$ and $BD \equiv B'D'$, then $AD \equiv A'D'$.

In itself a point-pair not only has no order, it does not even possess sense. But a sect, a point-pair on a straightest, has sense, and $AB \equiv BA$ must be proved or assumed. It may be assumed without introducing any divergence between the old concept, superposable, and the more fundamental concept, congruent. AB is superposed on BA by a semi-rotation about their common mid-point. On the sphere an angle, the figure two rays from the same initial point, has sense.

The analogue of the semi-rotation of a sect about its mid-point is the semi-rotation of an angle about its mid-ray.

If two figures have central symmetry in a plane, either can be made to coincide with the other by turning it in the plane through two right angles. This holds good when for "plane" we substitute "sphere." Any sect and its inverse, AB and BA, are such figures. They are symcentral about the mid-point of either.

If two figures have axial symmetry in a plane, they can be made to coincide by folding the plane over along the axis, but not by any sliding in their plane. Two axially symmetrical figures in a plane can be brought into coincidence by a semi-revolution of one about the axis. That is, we must use the third dimension of space, and then their congruence depends on the property of the plane that its two sides are indistinguishably alike, any plane will completely fit its trace after being turned over. This procedure, folding over along a line, can have no place in a strictly two-dimensional geometry.

So figures with axial symmetry on a sphere cannot be made to coincide. Let symmetric henceforth mean axially symmetric, and be denoted by $+$. A spherical angle and its inverse, $\widehat{\measuredangle}(h, k)$ and $\widehat{\measuredangle}(k, h)$ are not symcentral and cannot be brought into coincidence. Should we take as a definition:

An angle is called symmetric to another to whose inverse it is congruent; $\measuredangle(h, k) + \measuredangle(v, w)$ when $\measuredangle(h, k) \equiv \measuredangle(w, v)$, then from $\measuredangle(h, k) \equiv \measuredangle(h, k)$ comes $\measuredangle(h, k) + \measuredangle(k, h)$, but these two are not superposable, cannot be made to coincide.

To those then who have made ideal superposition the basis and test of con-

gruence, the fact that a spherical angle can in no way be placed upon its inverse introduces a radical difference between their presentation of spherics and the familiar presentations of plane geometry. For them many theorems would be bifurcate, for example, Theorem: Any two right angles are either congruent or symmetric. But since congruence in no way depends on the subsequenr concept, motion, nothing is more simple than to assume $\widehat{\chi}(h, k) \equiv \widehat{\chi}(k, h)$.

The three points which determine a triangle have no order and no betweenness, in the specific meaning, for that would make two of a like character and the third of a different character. But as a triad of sects, a triangle has as an individual 'umlaufssinn', tour-sense.

If the congruence of sects is connected with that of angles by a triangle assumption, then if this is restricted to triangles of the same tour-sense, it will not give us the congruence of the basal angles in an isosceles triangle. To attain this is in that case needful the adjunction of one or two continuity assumptions.

Thus we see that beyond the one dimensional space-symmetry implied in the congruence of sect and angle with their inverses, there is a quite distinct two-dimensional space symmetry. This is usually unrecognized in the ordinary treatment of plane geometry, since a plane triangle can have its tour-sense changed by turning it over in the third dimension.

In two-dimensional spherics this is impossible, so although this two-dimensional space-symmetry is presumed in the triangle-assumption's non-recognition of tour-sense, yet it is customary to note its perception in the distinction between congruent and symmetric triangles.

Had this distinction been retained further back, namely for angles, we might have used it in the definition: Two triangles are called symmetric when their corresponding sides are congruent and their corresponding angles are symmetric. But a definition perhaps more desirable comes from setting up the distinction of right side and left side of an angle.

Besides the assumptions of congruence, we need no metric assumptions, and as definition or axiom are well rid of that snaky phrase: "A straight line is the shortest distance between two points." In this reference see Georg Hamel: Ueber die Geometrieen, in denen die Geraden die Kürzesten sind, Math. Ann. Bd. 57, 1903.

Now as to continuity, there seems more call for such assumptions in spherics than in planimetry, since, for example, in the plane a sect may be readily divided into any desired number of parts, while in spherics recourse must be had to a continuity assumption even to show that a given sect has third parts, that one-third of a given sect exists. Nevertheless continuity remains costly. Even Hilbert has not succeeded in attaining a simple treatment of it. His 'Axiom der Vollständigkeit' and 'Axiom der Nachbarschaft' seem inelegant lumps in his beautiful and fine mosaic. Russell says (*Principles of Mathematics*, p. 440): "Whether the axiom of continuity be true as regards our actual space is a question which I see no means of deciding. "For any such question must be empirical, and it would be quite impossible to distinguish empirically what may be

called a rational space from a continuous space."

In my *Rational Geometry* I treat Non-Euclidean Spherics without any continuity assumption whatsoever.

PROBLEMS FOR SOLUTION.

ALGEBRA.

E. Kesner, Salida, Col., solved 208 and 209.

211. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Prove that $p-qx$ and $q-px$ tend to equality as x dimishes to zero, but yet that their limits are not equal. [Edwards' *Differential Calculus*, p. 7, ex. 10.]

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

The limits of $p-qx$ and $q-px$ as x dimishes to zero, are p and q. Suppose $p>q$, then by subtraction the difference at the limit is $p-q$ and the difference for any value of x is $p-q-qx+px$ which evidently decreases as x diminishes. Likewise for $q>p$. Therefore $p-qx$ and $q-px$ tend to equality as x diminishes to zero.

Also solved by G. B. M. Zerr, and J. Scheffer.

212. Proposed by F. P. MATZ, Ph. D., Sc. D.

Solve $\dfrac{x^2+2x+3}{x^2-2x+3}+\dfrac{x^2-2x+3}{x^2+2x+3}=\dfrac{10}{3}$.

I. Solution by O. S. WESTCOTT, Waller High School, Chicago, Ill., ELMER SCHUYLER, Brooklyn, N. Y., and J. SCHEFFER, Hagerstown, Md.

If we write $\dfrac{x^2+2x+3}{x^2-2x+3}=y$, then will $y+\dfrac{1}{y}=\frac{10}{3}$, and $y=3$ or $\frac{1}{3}$.

a) If $\dfrac{x^2+2x+3}{x^2-2x+3}=3$, $x=3$ or 1.

b) If $\dfrac{x^2+2x+3}{x^2-2x+3}=\frac{1}{3}$, $x=-3$ or -1.

II. Solution by A. H. HOLMES, Brunswick, Maine.

Clearing the given equation of fractions and reducing, we obtain $x^4-10x^2=-9$, whence $x=1, -1, 3, -3$.

Also solved by G. W. Greenwood, E. L. Rich, M. E. Graber, G. B. M. Zerr, F. D. Posey, S. S. Flory, L. E. Newcomb, and E. Kesner.

213. Proposed by F. P. MATZ, Ph. D., Sc. D.

Find the two roots of the equation $x^5-209x+56=0$, whose product is unity.

I. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill., and F. D. POSEY, A. B., San Mateo, Cal.

The equation may be written $(x^2-4x+1)(x^3+4x^2+15x+56)=0$. Hence two of the roots are $2\pm\sqrt{3}$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let y and $1/y$ be the roots, then

$$y^5-209y+56=0 \quad \ldots\ldots (1),$$
$$56y^5-209y^4+1=0 \quad \ldots\ldots (2).$$

56 times (1)$-$(2) gives $209y^4-56\times 209y+56^2-1=0$, or $209(y^4-56y+15)=0$.

$\therefore y^4-56y+15=0 \quad \ldots\ldots (3)$.

(1)-56 times (2) gives, after dividing by y, $15y^4-56y^3+1=0 \quad \ldots\ldots (4)$.

15 times (3)$-$(4) gives $y^3-15y+4=0 \quad \ldots\ldots (5)$.

(3)-15 times (4) gives $4y^3-15y^2+1=0 \quad \ldots\ldots (6)$.

4 times (5)$-$(6) gives $y^2-4y+1=0$. $\therefore y=2\pm\sqrt{3}$.

Hence the roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Also solved by A. H. Holmes, J. Scheffer, Lloyd Holsinger, L. E. Newcomb, and Elmer Schuyler.

214. Proposed by Editor EPSTEEN.

How many terms are there in the expansion of $(x_1+x_2+\ldots\ldots+x_n)^n$?

I. Solution by F. OWENS, M.A.; Mrs. F. OWENS, M.A.; G. B. M. ZERR, A. M., Ph. D.; G. W. GREENWOOD, M. A.

Let ${}_nu_r$ denote the number of terms in the rth power of an expression consisting of n terms. By considering it in the form

$$[x_1+(x_2+x_3+\ldots\ldots+x_n)]^r$$

we see that

$${}_nu_r=1+{}_{n-1}u_1+{}_{n-1}u_2+\ldots\ldots+{}_{n-1}u_r.$$

$\therefore {}_nu_r={}_nu_{r-1}+{}_{n-1}u_r$.

Hence we may write down the number of terms in any power of any expression thus:

1	1	1	1	1	.....
1	2	3	4	5	.....
1	3	6	10	15	.....
1	4	10	20	35	.....
....	....	....	....	....	.....

where each element is obtained by adding the one the left to the one above it.

It can be seen, from Pascal's Triangle, that the number of terms is given by the expression for the $(r+1)$st terms of the nth order of figurate numbers, ${}_nu_r=\frac{(n+r-1)!}{r!\,(n-1)!}$. In particular ${}_nu_n=\frac{(2n-1)!}{n!(n-1)!}$.

II. Solution by J. SCHEFFER, Hagerstown, Md.

The problem reduces to finding the number of products of n quantities and their power. By expanding, $\frac{1}{1-x_1x}$, $\frac{1}{1-x_2x}$, $\frac{1}{1-x_3x}$, etc., into series, we find,

$$\frac{1}{1-x_ix}=1+x_ix+x_i^2x^2+x_i^3x^3+\ldots\ldots \quad (i=1, 2, \ldots\ldots, n).$$

$\left(\frac{1}{1-x_1x}\right)\left(\frac{1}{1-x_2x}\right)\left(\frac{1}{1-x_3x}\right)\ldots\ldots=1+s_1x+s_2x^2+s_3x^3+\ldots\ldots$ where $s_1=x_1+x_2+\ldots\ldots x_n$, $s_2=x_1^2+x_2^2+x_3^2\ldots\ldots+x_1x_2+x_1x_3\ldots\ldots+x_2x_3\ldots\ldots$, $s_3=x_1^3+x_2^3+\ldots\ldots x_n^2x_1^2x_2+x_1^2x_3+\ldots\ldots+x_1x_2x_3+\ldots\ldots$

To find the number of products in any of these sets of products, we put $x_1=x_2=x_3=\ldots\ldots=x_n=1$, and we get

$$\frac{1}{1-x_1x}\cdot\frac{1}{1-x_2x}\cdot\frac{1}{1-x_0x}\ldots\ldots=\frac{1}{(1-x)^n}=(1-x)^{-n}.$$

$\therefore s_n$ is equal to the coefficients of x^n in the expression of $(1-x)^{-n}$.

$\therefore s_n=\frac{n(n+1)(n+2)\ldots\ldots[n+(n-1)]}{n!}$, or, multiplying numerator and denominator by $(n-1)!$ we get $\frac{(2n-1)!}{(n-1)!n!}$ as the required result.

Also solved by L. E. Newcomb, and Elmer Schuyler.

GEOMETRY.

Two solutions of 238 were received from L. E. Newcomb, Los Gatos, Cal.

239. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Divide the sides of a triangle ABC internally in P, Q, R, so that $BP/PC=CQ/QA=AR/RB$. QR cuts BC externally in S. Show that BS is to CS in the duplicate ratio of CP to PB.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Draw SD parallel to AB; let m=any proper fraction, and let $PC=ma$, $QA=mb$, $RB=mc$. Then

$$\frac{BP}{PC}=\frac{a(1-m)}{am}=\frac{CQ}{QA}=\frac{b(1-m)}{bm}=\frac{AR}{RB}=\frac{c(1-m)}{cm}=\frac{1-m}{m}.$$

Considering AB, AC as the axes of coördinates the equation to RQ is

$$\frac{x}{bm}+\frac{y}{c(1-m)}=1, \text{ equation to } BC \text{ is } \frac{x}{b}+\frac{y}{c}=1.$$

$$\therefore x=AD=\frac{bm^2}{2m-1},\ y=DS=\frac{c(1-m)^2}{2m-1}=cn.$$

$$CD=AD-b=\frac{b(1-m)^2}{2m-1}=bn.$$

$CS=n\sqrt{(b^2+c^2-2bc\cos A)}=an$ since $\angle CDS=A$.
$BS=a+CS=a(n+1)$.

$$\therefore CS=\frac{a(1-m)^2}{2m-1},\ BS=\frac{am^2}{2m-1}.\quad \therefore BS:CS=m^2:(1-m)^2.$$

$CP^2=a^2m^2$, $BP^2=a^2(1-m)^2$.
$\therefore CP^2:BP^2=m^2:(1-m)^2$, $\therefore BS:CS=CP^2:BP^2$.

II. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Through C draw a parallel to RQ which will intersect AB in a point M, say, then

$$\frac{BS}{CS}=\frac{RB}{MR}.\ \text{But}\ \frac{MR}{CQ}=\frac{AR}{QA}.\ \therefore \frac{BS}{CS}=\frac{RB}{AR}\cdot\frac{QA}{CQ}=\left(\frac{PC}{PB}\right)^2.$$

III. Solution by J. SCHEFFER, Hagerstown, Md.

Draw CF parallel to AB, meeting RQ in T.

From $\frac{PB}{PC}=\frac{QC}{QA}=\frac{RA}{RB}$, we have $PB.QA=PC.QC$, $PB.RB=PC.RA$.

By multiplication, $PB^2.QA.RB=PC^2.QC.RA$, or $\frac{PB^2}{PC^2}=\frac{QC.RA}{QA.RB}$ .... (1).

But $\frac{RA}{CT}=\frac{QA}{QC}$, and $\frac{CT}{RB}=\frac{CS}{BS}$; $\therefore \frac{RA}{RB}=\frac{QA}{QC}\cdot\frac{CS}{BS}$.

Substituting in (1), $\frac{PB^2}{PC^2}=\frac{CS}{BS}$.

240. Proposed by B. F. BURLESON.

The points P_1, P_2, P_3 in the perpendicular BD of an isosceles triangle with equal angles at A and C are at the intersection of the three perpendiculars of the triangles, the center of its inscribed, and the center of its circumscribed circles, respectively. The distance from P_1 to P_2 is $m=16$ rods, and the distance from P_2 to P_3 is $n=17$ rods. Required the radii R and r of the triangle's circumscribed and inscribed circles, the perpendiculars $BD=P_b$, the base $AC=2b$, and one of the equal sides as $AB=a$.

I. Solution by the PROPOSER.

We derive from Chauvenet's *Trigonometry*, equation 300 and equation 298,

$$R^2-2Rr=n^2 \text{.....(1)},$$
$$R^2-4r^2=(m+n)^2-2m^2 \text{.....(2)}.$$

Resolving (1) and (2), we find

$$R=\frac{n^2}{n-m}=289 \text{ rods, and } r=\frac{m(2n-m)}{2(n-m)}=144 \text{ rods.}$$

We have in any isosceles triangle

$$P_b=\frac{2rb^2}{b^2-r^2}=\frac{4n^2-m^2}{2(n-m)}=450 \text{ rods.....(3).} \qquad \text{Also } P_b=\sqrt{[(R^2-b^2)+R]}\text{....(4).}$$

Equating (3) and (4) and solving, we find that

$$2b=AC=2\sqrt{\{r[2R-r-2\sqrt{(R^2-2Rr)}]\}}=\frac{n}{n-m}\sqrt{(4n^2-m^2)}=480 \text{ rods.}$$

Whence $a=AB=\sqrt{(P_b^2+b^2)}=\frac{n}{n-m}\sqrt{(4n^2-m^2)}=510$ rods.

II. Solution by A. H. HOLMES, Brunswick, Me.

We have $BD=\sqrt{(a^2-b^2)}$, $r=\frac{b\sqrt{(a^2-b^2)}}{a+b}$, and $R=\frac{a^2}{2\sqrt{(a^2-b^2)}}$,

$$DP_1=\frac{b^2}{\sqrt{(a^2-b^2)}},\ DP_2=r, \text{ and } DP_3=\frac{a^2-2b^2}{2\sqrt{(a^2-b^2)}}.$$

$$\therefore \frac{b\sqrt{(a^2-b^2)}}{a+b}-\frac{b^2}{\sqrt{(a^2-b^2)}}=m=16 \text{.....(1)},$$

$$\frac{a^2-2b^2}{2\sqrt{(a^2-b^2)}}-\frac{b\sqrt{(a^2-b^2)}}{a+b}=n=17 \text{.....(2)}.$$

From (1), $\sqrt{(a^2-b^2)}=\frac{b}{m}(a-2b)$.

From (2), $\sqrt{(a^2-b^2)}=\frac{a}{2n}(a-2b)$. $\quad\therefore a=\frac{2n}{m}b$.

Putting the value of a in (1), $b=\frac{m}{2(n-m)}\sqrt{(4n^2-m^2)}$.

$$\therefore a=\frac{n}{n-m}\sqrt{(4n^2-m^2)},\ BD=\frac{4n^2-m^2}{2(n-m)},\ r=\frac{m\,(2n-m)}{2(n-m)},\ R=\frac{n^2}{n-m}.$$

When $m=16$ and $n=17$, the equal sides become 510, base=480, perpendicular=450, $r=144$, $R=289$.

Also solved by G. B. M. Zerr, J. Scheffer, L. E. Newcomb.

241. Proposed by Editor EPSTEEN.

If two conics have each double contact with a third, their chords of contact with that conic, and two of the lines through their common points, will meet in a point and form a harmonic pencil.

I. Solution by F. D. POSEY, A. B., San Mateo, Cal.

Let $S=0$ be the equation of the conic which has double contact with the first conic in the line $\alpha_1=0$ and with the second conic in $\alpha_2=0$. The equation of these two conics may then be written

$$c-\lambda_1^2\alpha_1^2=0 \text{.....(1)}, \quad c-\lambda_2^2\alpha_2^2=0 \text{.....(2)}.$$

The two straight lines $\lambda_1^2\alpha_1^2-\lambda_2^2\alpha_2^2=0$ go through the common points of (1) and (2), through the intersections of $\alpha_1=0$ with $\alpha_2=0$, and it is evident that the lines $\alpha_1=0$, $\lambda_1\alpha_1-\lambda_2\alpha_2=0$, $\alpha_2=0$, $\lambda_1\alpha_1+\lambda_2\alpha_2=0$, form a harmonic pencil.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $S=0$ be the equation to the third conic. Then $S=k\alpha^2$, $S=k\beta^2$ are the equations to the other two conics (Salmon's *Conic Sections*, Art. 252, page 234). Subtracting, $\alpha^2-\beta^2=0$ or $(\alpha-\beta)(\alpha+\beta)=0$. This represents a pair of chords of intersection passing through the intersection of the chords of contact $\alpha=0$, $\beta=0$. Also $\alpha=0$, $\beta=0$, $\alpha-\beta=0$, $\alpha+\beta=0$ form a harmonic pencil. (See Salmon's *Conic Sections*, Art. 263, page 242).

Also solved by G. W. Greenwood.

242. Proposed by the late MARCUS BAKER.

In a trapezoid $ABCD$, upper base $BC=a$, lower base $AD=b$, a line CP is drawn from vertex C to any point P in the base AD, such that $PD=mb$. The line CP intersects the diagonal BP in M and MN is drawn parallel to the bases meeting CD in N; then is $MN=\dfrac{abm}{a+bm}$.

Solution by the PROPOSER, and G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

From the similar triangles DMN, DBC, we have

$$\frac{MN}{a}=\frac{MD}{BD}.$$

From the similar triangles MPD, MBC, we have

$$\frac{MD}{BD}=\frac{mb}{a+mb}. \qquad \therefore MN=\frac{amb}{a+mb}.$$

Also solved by A. H. Holmes, G. B. M. Zerr, F. D. Posey, J. Scheffer, and L. E. Newcomb.

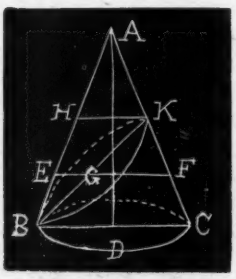
CALCULUS.

185. Proposed by A. H. HOLMES, Brunswick, Maine.

Required the perpendicular height of a right cone, radius of base being unity, such that the maximum ellipse that can be cut from the cone shall equal the base of the cone in area.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $BC=2=c$; a, b the semi-axes of the ellipse, $\angle KBC=\theta$, $\angle DAC=\beta$. Then $BK=2a=c\cos\beta/\cos(\theta-\beta)$ or $a=\frac{1}{2}c\cos\beta/\cos(\theta-\beta)$; $b^2=EG.GF=\frac{1}{4}cHK$. But $HK=2a\cos(\theta+\beta)/\cos(\theta-\beta)$.



$\therefore b=\frac{1}{2}c\sqrt{[\cos(\theta+\beta)/\cos(\theta-\beta)]}$, and therefore $\pi ab=\frac{1}{4}\pi c^2\cos\beta\sqrt{[\cos(\theta+\beta)]}/[\cos(\theta-\beta)]^{\frac{1}{2}}$ = maximum.

$\therefore \dfrac{\sqrt{[\cos(\theta+\beta)]}}{\cos(\theta-\beta)\sqrt{[\cos(\theta-\beta)]}}$ = maximum, and $\sin2\theta=2\sin2\beta$, or, $\sin\theta=\frac{1}{2}\sqrt{\{2[1+\sqrt{(1-4\sin^2 2\beta)}]\}}$.

Since $c=2$ and $\pi ab=\pi c^2/4$, we have $\cos^2\beta\cos(\theta+\beta)=\cos^3(\theta-\beta)$. Now $\cos\theta=\frac{1}{2}\sqrt{\{2[1-\sqrt{(1-4\sin^2 2\beta)}]\}}$,

$$\cos^3\beta\sin\theta\cos\theta=4\sin\beta\cos^2\beta\cos^2\theta+3\sin^2\beta\cos\beta\sin\theta\cos\theta+\sin\beta\sin^2\theta.$$

Substituting the values of $\sin\theta$, $\cos\theta$ and reducing,

$$4\sin^2 2b+1=(2\cos2\beta+1)\sqrt{(1-4\sin^2 2\beta)}.$$

$\therefore 16\sin^4 2\beta+40\sin^2 2\beta=7$, and $\sin2\beta=\frac{1}{2}\sqrt{(4\sqrt{2}-5)}$; $\beta=11^\circ\ 57'\ 9''$.

$\therefore$ Altitude $=\cot\beta=4.72388$.

Also solved by J. Scheffer, the Proposer, and L. E. Newcomb.

186. Proposed by Editor EPSTEEN.

Evaluate $\displaystyle\int_0^\infty \frac{\sin my}{y}dy$, $\displaystyle\int_0^\infty \frac{\cos my}{y}dy$.

I. Solution by IDA M. SCHOTTENFELS, M. A., New York City.

$\displaystyle\int_0^\infty \frac{\sin my}{y}dy=+\tfrac{1}{2}\pi,\ 0,\ -\tfrac{1}{2}\pi$, according as $m>0$, $m=0$, $m<0$ if $y>0$.

Proof. If $y>0$ then $\displaystyle\int_0^\infty e^{-yx}dx=1/y$. Multiplying both members by $\sin my$:

(1) $\displaystyle\sin my\int_0^\infty e^{-yx}dx=\frac{\sin my}{y}$; (2) $\displaystyle\int_0^\infty e^{-yx}\sin my\,dx=\frac{\sin my}{y}$.

Integrating (2) with respect to y between 0 and ∞,

$$(3) \quad \int_0^\infty \left(\int_0^\infty e^{-yx}\sin my\, dx\right)dy = \int_0^\infty \frac{\sin my}{y}dy.$$

Changing the order of integration since the limits are the same we have

$$\int_0^\infty \left(\int_0^\infty e^{-yx}\sin my\, dy\right)dx = \int_0^\infty \frac{\sin my}{y}dy.$$

But

$$(5) \quad \int_0^\infty e^{-yx}\sin my\, dy = \frac{m}{m^2+x^2} \quad (y>0).$$

The result in (5) is obtained by finding the indefinite integral and passing to the limit thus,

$$\int e^{-yx}\sin my\, dy = -e^{-yx}\left[\frac{m\cos my + x\sin my}{x^2+m^2}\right]$$

$$\lim_{y \doteq \infty} -e^{-yx}\left[\frac{m\cos my + x\sin my}{x^2+m^2}\right] = \frac{m}{x^2+m^2} \quad (y>0)$$

$$(6) \quad \int_0^\infty \frac{m}{x^2+m^2}dy = \int_0^\infty \frac{\sin my}{y}dy.$$

$$(7) \quad \int_0^\infty \frac{m}{x^2+m^2}dx = \tan^{-1}\frac{x}{m}\Big]_0^\infty$$

$\tan^{-1}\frac{x}{m}\Big]_0^\infty = \frac{1}{2}\pi,\ 0,\ -\frac{1}{2}\pi$ according as $m>0,\ m=0,\ m<0$, and therefore $\int_0^\infty \frac{\sin mx}{x}dx$ has the same values and equation (1) is true.

We now regard the given integral as a function of the parameter m thus

$$\int_0^\infty \frac{\sin my}{y}dy = \varphi(m)$$

and trace the resulting curve.

For all negative values of m the integral $\int_0^\infty \frac{\sin mx}{x}dx$ describes a line parallel to the m axis and at a distance $-\frac{1}{2}\pi$ from it until m reaches the value 0; here the function has a finite discontinuity and jumps from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, and for positive values of m continues in a straight line parallel to the m axis and at a distance $+\frac{1}{2}\pi$ from it.

In a similar manner $\int_0^\infty \frac{\cos my}{y}dy = \infty$ for real finite values of m.

II. Solution by S. A. COREY, Hiteman, Iowa.

We have $1/y=\int_0^\infty e^{-ay}da$ if $y>0$. Hence

$$\int_0^\infty \frac{\sin my}{y}dy=\int_0^\infty \sin my\int_0^\infty e^{-ay}da\,dy=\int_0^\infty\int_0^\infty \sin my\, e^{-ay}\,da\,dy,$$

$$=\int_0^\infty\int_0^\infty \sin my\, e^{-ay}\,dy\,da=\int_0^\infty \frac{mda}{a^2+m^2}$$

$=\frac{1}{2}\pi$ if $m>0$, $=-\frac{1}{2}\pi$ if $m<0$, $=0$ if $m=0$.

(See Byerly's *Integral Calculus*, page 100.) Similarly,

$$\int_0^\infty \frac{\cos my}{y}dy=\int_0^\infty \cos my\int_0^\infty e^{-ay}\,da\,dy=\int_0^\infty \frac{ada}{a^2+m^2}=0,$$

for all finite real values of m.

III. Remark by F. D. POSEY, A. B., San Mateo, Cal.

A discussion of the first of these integrals will be found on page 271, Art. 285 of Todhunter's *Integral Calculus*. The result there obtained is $\frac{1}{2}\pi$ if m be positive, $-\frac{1}{2}\pi$ if m be negative.

Also solved by G. W. Greenwood, G. B. M. Zerr, and J. Scheffer.

MISCELLANEOUS.

146. Proposed by F. P. MATZ, Ph. D., Sc. D.

Given $\left.\begin{matrix} a\cos\alpha+b\sin\alpha=c \\ a\cos\beta+b\sin\beta=c \end{matrix}\right\}$ to prove that

$$\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2},\text{ and }\cot\alpha+\cot\beta=\frac{2ab}{c^2-a^2}.$$

I. Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

We have at once $\dfrac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}=-b/a$ ..... (1).

Now $\cos\alpha-\cos\beta=-2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$ and $\sin\alpha-\sin\beta=2\cos\frac{1}{2}(\alpha+\beta)\times\sin\frac{1}{2}(\alpha-\beta)$. Substituting these values in (1) we obtain $\tan\frac{1}{2}(\alpha+\beta)=b/a$. Since $\tan\frac{1}{2}(\alpha+\beta)=\pm\sqrt{\dfrac{1-\cos(\alpha+\beta)}{1+\cos(\alpha+\beta)}}=b/a$, and solving,

$$\cos(\alpha+\beta)=\frac{a^2-b^2}{a^2+b^2}\text{.....(2)};\ \sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}\text{.....(3)}.$$

Multiplying together the given equation and substituting the values of $\cos(\alpha+\beta)$,

$\sin(\alpha+\beta)$ from (2) and (3) we find that $\sin\alpha\sin\beta=\frac{c^2-a^2}{a^2+b^2}$; and therefore

$$\cot\alpha+\cot\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}=\frac{2ab}{c^2-a^2}.$$

II. Solution by L. S. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

Subtracting the second equation from the first, $a(\cos\alpha-\cos\beta)+b(\sin\alpha-\sin\beta)=0$. Hence, $\sin\frac{1}{2}(\alpha-\beta)[a\sin\frac{1}{2}(\alpha+\beta)-b\cos\frac{1}{2}(\alpha+\beta)]=0$. Either $\sin\frac{1}{2}(\alpha-\beta)=0$ or $a\sin\frac{1}{2}(\alpha+\beta)=b\cos\frac{1}{2}(\alpha+\beta)$. In the first case we have the trivial solution $\alpha=\beta$. In the second case, we put $(\alpha+\beta)=x$, $\sin\frac{1}{2}x=b/a\cos\frac{1}{2}x$. Since $\sin x=2\sin\frac{1}{2}x\cos\frac{1}{2}x$ the latter equation takes the form $(a^2+b^2)\sin^2x=2ab\sin x$, whence, excluding the value $\sin x=0$ introduced by squaring, $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$.

Secondly, $\cot\alpha+\cot\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$. From the first of the given equations $a\sqrt{(1-\sin^2\alpha)}+b\sin\alpha=c$. Solving for $\sin\alpha$, we have $\sin\alpha=\frac{bc\pm a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2}$

Similarly, solving the second of the given equations, $\sin\beta=\frac{bc\pm a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2}$

Since the solution $\alpha=\beta$ was rejected, $\sin\alpha\sin\beta=\frac{c^2-a^2}{a^2+b^2}$.

Hence $\cot\alpha+\cot\beta=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$ becomes equal to $\frac{2ab}{c^2-a^2}$.

III. Solution by A. H. HOLMES, Brunswick, Maine.

From $a\cos\alpha+b\sin\alpha=c$,

$$\sin\alpha=\frac{bc+a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2},\quad \cos\alpha=\frac{ac-b\sqrt{(a^2+b^2-c^2)}}{a^2+b^2},$$

and from $a\cos\beta+b\sin\beta=c$,

$$\sin\beta=\frac{bc-a\sqrt{(a^2+b^2-c^2)}}{a^2+b^2},\quad \cos\beta=\frac{ac+b\sqrt{(a^2+b^2-c^2)}}{a^2+b^2}.$$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$. Putting for $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$ their values above, and reducing, $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$, $\cot\alpha+\cot\beta=\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}$.

Introducing the values of these sines and cosines, and reducing, $\cot\alpha+\cot\beta=\frac{2ab}{c^2-a^2}$.

Also solved by G. B. M. Zerr, G. W. Greenwood, J. Scheffer, and E. L. Rich.

PROBLEMS FOR SOLUTION.

ALGEBRA.

219. Proposed by Dr. SAUL EPSTEEN, The University of Chicago.

Sum to infinity $\frac{1.2}{3}+\frac{2.3}{3^2}+\frac{3.4}{3^3}+\ldots\ldots$

220. Proposed by L. ROBINSON, B. S., Philadelphia, Pa.

Find the sum of the first $n+1$ terms of the series

$$1+\frac{m}{1!}+\frac{m(m+1)}{2!}+\frac{m(m+1)(m+2)}{3!}+\ldots\ldots$$

221. Proposed by F. P. MATZ, Ph. D., Sc. D.

Eliminate the unknowns from

$$x/y+y/z+z/x=a \ldots\ldots(1), \quad x/z+y/x+z/y=b \ldots\ldots(2),$$
$$(x/y+y/z)(y/z+z/x)(z/x+x/y)=c \ldots\ldots(3).$$

222. Proposed by G. W. WALKER, Camden, N. J.

Extract the square root of $87-12\sqrt{42}$.

GEOMETRY.

245. Proposed by J. H. M. MACLAGAN-WEDDERBURN, M. A., Chicago, Ill.

Given two lines of fixed length, $AB=a$, $BC=b$, perpendicular to each other. A line CP is drawn making $\angle BPC=\theta$. AD is the perpendicular to AB meeting CP in D. Find by Euclidean construction the angle θ such that $AD^2\cos^2\theta+b^2\sin^2\theta$ is a minimum.

246. Proposed by T. L. CROYES, Paris, France.

Given a movable point O on a fixed diameter of a circle S, an inscribed triangle ABC, and the perpendiculars OM, ON, OP from the point O on the sides AB, AC, BC. Prove, by pure geometry, that the circle circumscribing the triangle MNP will always pass through a fixed point.

247. Proposed by SETH PRATT, C. E., Tecumseh, Neb.

From two given points without a circle to draw two lines meeting in the circumference and making equal angles with the tangent at that point.

AVERAGE AND PROBABILITY.

131. Proposed by F. P. MATZ, Ph. D., Sc. D.

In a given square an arc is described at random the center being one of the vertices of the square. What is the probability that this arc is longer than a side of the square?

NOTE.—Problems and solutions in the departments of Geometry, Calculus, Mechanics, and Average and Probability should be sent to B. F. Finkel; and those in the departments of Algebra, Diophantine Analysis, Miscellaneous, and Group Theory should be sent to Dr. Saul Epsteen. Our contributors should carefully observe this notice if proper credit for contributions is to be given.

NOTES.

Mr. J. R. Lucky has been appointed assistant in mathematics at Oberlin College.

Mr. R. G. D. Richardson has been appointed instructor in mathematics in Yale University.

Dr. K. Schmidt has been appointed professor of mathematics in the University of Florida.

Prof. Alois F. Kovarik has been appointed instructor in physics in the University of Minnesota.

The first part of the French edition of the *Encyklopädie der Mathematischen Wissenschaften* has appeared.

Dr. Harvey C. DeMotte, professor of mathematics in the Illinois Wesleyan University, died on December 16, 1904.

Dr. H. B. Evans has been promoted to an assistant professorship in mathematics in the University of Pennsylvania.

Prof. F. S. Luther, professor of mathematics in Trinity College, Hartford, Conn., has been elected president of the institution.

The editors are indebted to Mr. H. B. Leonard for assistance in compiling the index of Vol. XI of the MONTHLY, which will appear with the February number.

Dr. J. Stebbins has been promoted to an assistant professorship in mathematics and Mr. A. H. Wilson has been appointed to an instructorship in the University of Illinois.

Dr. R. S. Woodward, professor of mechanics and mathematical physics, and Dean of the School of Pure Science, Columbia University, has been elected president of the Carnegie Institution of Washington, Dr. Gillman having resigned on his 70th birthday.

Dr. F. H. Safford, of the University of Pennsylvania, has recently completed an exhaustive discussion of the seating of seven persons at a round table (Problem 99, March, 1899, March, 1900, and April, 1904). All possible arrangements have been tabulated, with no assumptions concerning groups, and all solutions found have been reduced to a single form, the one previously given.

A mathematical section of the California Teachers' Association was organized on December 26, 1904, at San Jose. G. A. Miller, Stanford University, was elected as president, and J. F. Smith, Campbell High School, as Secretary. The main object of the Association is to arouse more interest in mathematical pedagogy by means of separate meetings for the discussion of recent mathematical movements.

David Eugene Smith, professor of mathematics in Teachers' College, Columbia University, addressed the students of the State Normal School at Plattsburg, N. Y., on January 19, 1905. The subject of his lecture was the History of Mathematics. Professor Smith's recent monograph, "The Outlook for Arithmetic in America," is a valuable contribution to pedagogical literature. As a practical exposition of the author's theories, his two new arithmetics (Ginn & Co., publishers) are attracting wide attention.

BOOKS.

An Introduction to Projective Geometry and Its Applications. An Analytic and Synthetic Treatment. By Arnold Emch, Ph. D., Professor of Graphics and Mathematics in the University of Colorado. 8vo. Cloth, vii+267 pages, 114 figures. Price, $2.50. New York: John Wiley & Sons.

In contrast to the usual presentation of projective geometry in a purely systematic form with little attention given to applications, the present book treats the subject with a view to utility, considerable space being given to practical applications.

In addition to the usual subjects treated in elementary treatises, two chapters on pencils and ranges of conics, including cubics, and on applications of mechanics have been added. As an example of the power of projective geometry, the Stinerian Transformation of the Cubic, treated in Chapter IV, may be cited, and as a particular novel feature of the work, the realization of collineations by linkages, described in Chapter V, may be mentioned. A departure from fairly well established conventional notations occasionally occurs. For example, page 172, the conics whose equations are $u = 0$ and $u_1 = 0$ are referred to as the conics U and U_1. It seems to me to be better to refer to the geometric entities by their algebraic representatives, viz. u and u_1, thus avoiding the danger of taking U and U_1 to be different from u and u_1 when they are intended to be the same.

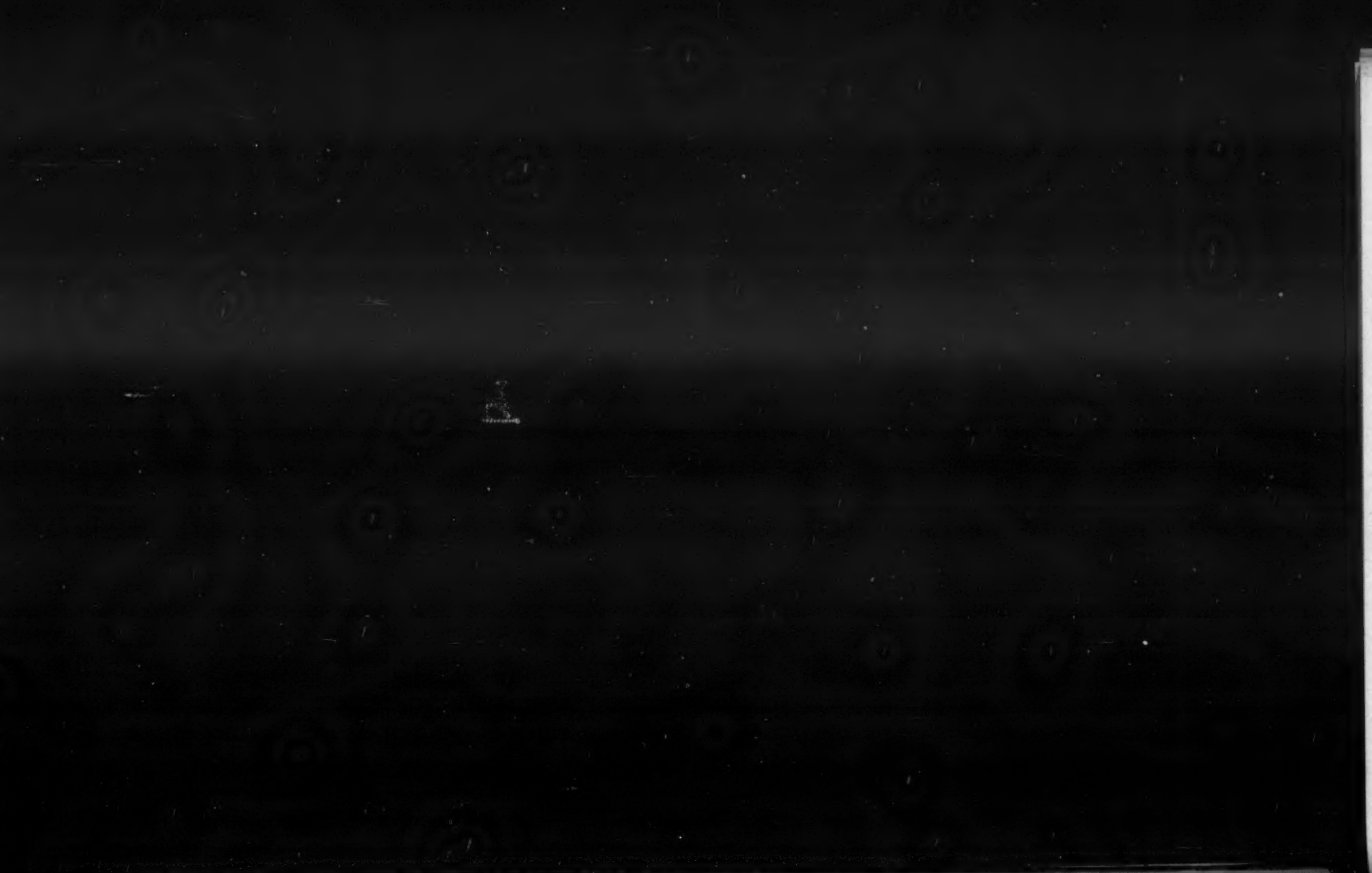
However, the whole subject as treated by the author is unusually clear and well adapted to the needs of the student of mathematics as well as to the practical mathematician. B. F. F.

The Foundations of the Euclidean Geometry as Viewed from the Standpoint of Kinematics. A Dissertation submitted to the Board of University Studies of the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy. By I. E. Rabinovitch. 8vo. Paper Cover, 116 pages. Published by the author.

The first thirty-seven pages of this thesis is devoted to a resumé and comparison of the researches of the various noted mathematicians who have considered the subject of Non-Euclidean Geometry. The remaining part of the thesis is devoted to the investigations of the Foundations of Geometry from the kinematical standpoint. This thesis is certainly a very valuable contribution to the science of mathematics. B. F. F.

The Elements of Analytical Geometry. By Percy F. Smith, Ph. D., Professor of Mathematics in the Scheffield Scientific School of Yale University, and Arthur S. Gale, Ph. D., Instructor in Mathematics in Yale College. 8vo. Cloth, xii+424 pages. Price, $2.00. Boston and Chicago: Ginn & Co.

A glance at the table of contents of this volume will show that a number of topics not usually found in elementary treatises are here discussed, some of which are the invariant properties of the equation of the second degree and Inversion. The whole subject is admirably treated. B. F. F.



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THE AMERICAN MATHEMATICAL MONTHLY.

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SOME NEW RATIOS OF CONIC CURVES.

By ALAN S. HAWKESWORTH.

[Concluded from January Number.]

Theorem 9. The focal chords through corresponding points [p and P] upon reciprocal curves determine at their other extremities two other corresponding points [q and Q]; and are thus corresponding chords [Figure 2].

Let p on the ellipse, and P on the hyperbola be the corresponding points, determined by the common radius $S'pEP$. And let pfq and PSQ be the respective focal chords. Then will qQ be also corresponding points, and have the determining radius $S'qFQ$ in common.

For, angles pfS' and PSd are equal [Theorem 6]; and hence also angles qfS' and QSd; or qfa and QSA; which angles must therefore subtend corresponding points [Theorem 6, Corollary 1], upon a common determining radius $S'qFQ$. And similarly, if $p'fq'$ and $P'Q'S$ be the corresponding focal chords; they are determined by the common radius $S'q'Q'$, and diameter $p'S'P'$ [Figure 3]. While, if focal chords through S' are chosen, still more obviously is this true; for the said focal chords and their determining radius or diameter now coincide.

Corollary 1. If lines Ef and Ff, ES and FS be bisected in rr' and RR' respectively, by the several auxiliary circles [Theorem 2], then these points $rr'R'R$ form a rhomboid. For rR and $r'R'$ are fixed, in both magnitude and direction, being parallel to, and one half of fS [Theorem 6, Corollary 3]; while EF, the director chord, which is similarly parallel to, and double of rr', and RR', is clearly variable, both in length and direction.

Theorem 10. The three tangents of the three corresponding points upon